Finite Difference Solution for Transient Cooling of a Radiating-Conducting Semitransparent Layer

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Transient solutions were obtained for cooling a semitransparent material by radiation and conduction. The layer is in a vacuum environment so the only means for heat dissipation is by radiation from within the medium leaving through the boundaries. Heat conduction serves only to partially equalize temperatures across the layer. As the optical thickness is increased, steep temperature gradients exist near the boundaries when conduction is relatively small. A solution procedure is required that will provide accurate temperature distributions adjacent to the boundaries, or radiative heat losses will be in error. The approach utilized numerical Gaussian integration to obtain the local radiative source term, and a finite difference procedure with variable space and time increments to solve the transient energy equation.

Nomenclature

a = absorption coefficient of layer
c_p = specific heat of radiating medium
D = thickness of radiating layer
$E_1, \ldots, E_n$ = exponential integral functions,

$$E_n(x) = \int_0^1 \mu^{n-2} \exp(-x/\mu)\mathrm{d}\mu$$

k = thermal conductivity of radiating medium
N = conduction-radiation parameter, $k/4\sigma T_i^3 D$
q_r = radiative heat flow per unit area and time
R = abbreviation for radiation terms in energy equation
T = absolute temperature
T_e = temperature of surrounding environment
T_i = initial temperature of radiating layer
T_m = integrated mean temperature
t = dimensionless temperature, T/T_i
X = dimensionless coordinate, x/D
x = coordinate in direction across layer
ε_m = emittance of layer based on instantaneous value of T_m
ε_{ut} = emittance for a layer at uniform temperature
θ = time
κ = optical coordinate, ax
κ_D = optical thickness of layer, aD
ρ = density of radiating medium
σ = Stefan-Boltzmann constant
τ = dimensionless time, $(4\sigma T_i^3/\rho c_p D)\theta$

Subscripts
i = initial condition; the ith x location
n = at the nth time increment
ut = uniform temperature condition

Introduction

TRANSIENT thermal processes in semitransparent materials arise in applications such as using ceramic components and coatings at high temperatures, the formation of crystals by solidification in an outer space environment, and in windows in high-temperature devices. Radiative transport within the material acts simultaneously with heat conduction. Since radiation fluxes depend strongly on temperature level, accurate temperature distributions must be obtained, or the transient solution will yield erroneous results as time advances. For some types of thermal boundary conditions, approximate analytical methods, such as those related to radiative diffusion, may not deal with the boundary conditions with sufficient accuracy. The solution requires two operations. One is the integration of the contribution of the instantaneous temperature distribution surrounding each location to obtain the radiative source distribution within the medium. The second is the transient solution of the energy equation. A Gaussian integration method is used here to obtain the source function distribution, and an implicit finite difference procedure with variable grid size is used for solving the energy equation. This approach demonstrates that direct numerical procedures can be readily used to yield accurate transient cooling solutions for combined radiation and conduction where the surface temperature is unknown and varies with time.

The transient cooling of plane layer geometries has been examined in the literature for a variety of situations with single and multiple layers.[1-5] A common boundary condition is to specify the layer surface temperatures. In the present situation the radiating layer is cooling by exposure to a cold environment, and the surface temperatures are unknown functions of time. The environment is a vacuum, such as in outer space, and hence there is no means to remove energy from the layer surfaces by convection or conduction. The layer is semitransparent so that radiant energy from within the region passes out through its boundaries. This is volume emission so there is no radiation emitted from the surface itself which has no volume. Energy can be conducted into an elemental layer adjacent to the surface, but energy cannot be radiated exactly from the surface. Hence, in the absence of external conduction or convection, the internal conduction must decrease to zero as the surface is approached. It follows that the boundary conditions for the energy equation are that the boundaries in a vacuum environment are externally insulated with regard to the heat conduction portion of the energy transfer, and the normal temperature gradient at each surface is zero. Heat conduction redistributes energy within the layer, but energy is lost only by means of internal radiation passing through the surface.

For some conditions, such as for an optically thick medium, the transient temperature distributions become steep near a

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boundary because the radiative loss is mostly from a thin region adjacent to the surface. If the temperatures near the boundary are inaccurate, the radiative loss can be significantly in error. For a valid transient solution for the present external conditions, the zero temperature gradient condition at the boundary must be accurately achieved by the numerical procedure. Otherwise the solution will behave as if there is an additional energy loss or gain at the boundary, and this leads to an accumulative error in the overall heat balance during the transient cooling calculations. To obtain accurate temperature distributions near a boundary under conditions of transient radiative loss from the interior, a finite difference procedure was used with a variable increment size to concentrate grid points in the near-boundary regions.

The layer temperature is initially uniform and hence its initial emittance is for that condition. At the onset of transient cooling the near-boundary regions cool rapidly. This reduces the layer emittance since much of the radiation loss is originating from a volume region that has lower temperatures than are characteristic of the layer interior. For an initial period, the emittance continues to decrease with time; the extent of the decrease depends on the optical thickness. Then the radiation contribution becomes small because the temperature level has decreased, and the magnitude of conduction relative to radiation increases. This tends to make the temperature distribution become uniform as the transient proceeds further. The layer emittance then increases toward its initial value, which was for a uniform temperature layer.

The variation of emittance throughout the transient depends on the optical thickness of the medium, and on the initial conduction-radiation parameter, which contains the thermal conductivity. For certain conditions, such as an optically thin medium, the temperature distributions tend to be rather uniform throughout the entire transient. In this instance the use of the emittance corresponding to a uniform temperature distribution provides a very good approximation throughout the transient solution. The analysis shows for what ranges of parameters this is a good approximation, thus providing a simple solution.

Analysis

Energy Relations for Transient Cooling

A plane layer of thickness D, as shown in Fig. 1, consists of a gray emitting, absorbing, and nonscattering medium that is heat conducting. The layer is initially at a uniform temperature T_i and is then placed in much cooler surroundings so that energy is lost by radiation. The surroundings are a vacuum so radiation is the only mode for energy loss. The surrounding temperature T_e is low enough so that radiation from the surroundings to the layer can be neglected. The layer conducts heat internally, but because of the vacuum surroundings, there is no mechanism by which heat can be conducted or convected away from the boundaries. Hence as explained previously, $\partial T/\partial x$ is zero at both boundaries. The only means for energy loss is by radiation passing through the boundaries from within the medium. The energy equation has the form[6]

$$\rho c_p \frac{\partial T}{\partial \theta} = k \frac{\partial^2 T}{\partial x^2} - \frac{\partial q_r}{\partial x} \tag{1}$$

The gradient of the radiative flux is given in terms of $T(x, \theta)$ by Ref. 6:

$$\frac{\partial q_r}{\partial x} = 4a\sigma T^4 - 2a \int_0^D \sigma T^4(x^*, \theta) E_1(a|x - x^*|)\mathrm{d}x^* \tag{2}$$

Equations (1) and (2) are combined and placed in dimensionless form to yield

$$\frac{\partial t}{\partial \tau} = N \frac{\partial^2 t}{\partial X^2} - \kappa_D \left[t^4 - \frac{\kappa_D}{2} \int_0^1 t^4 E_1(|X - X^*|)\mathrm{d}X^* \right] \tag{3}$$

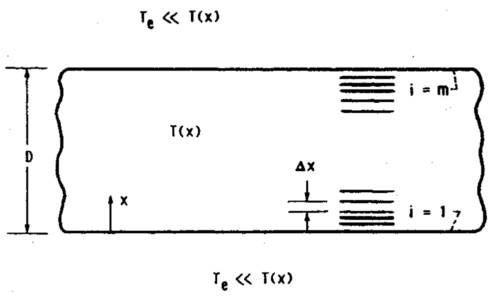


Fig. 1 Geometry and nomenclature for plane layer.

As discussed later, a finite difference procedure with variable increment sizes in both space and time is used to obtain the transient solution for $t(X, \tau)$. The initial condition for the layer at uniform temperature is $t(X, 0) = 1$. As discussed earlier, the boundary conditions are $\partial t/\partial X = 0$ at $X = 0,1$ for all τ. In the limit of radiation only, the $k(\partial^2 T/\partial x^2)$ term is not in the energy equation and the boundary conditions are not needed.

The numerical solution given later yields the transient temperature distribution in the layer. A quantity of interest is the transient emittance of the layer. The emittance is defined on the basis of a characteristic temperature; the instantaneous integrated mean temperature is used as it is a quantity of physical interest

$$T_m(\theta) = \frac{1}{D} \int_0^D T(x, \theta)\mathrm{d}x; \quad t_m(\tau) = \int_0^1 t(X, \tau)\mathrm{d}X \tag{4}$$

The emittance is defined such that the instantaneous heat dissipation from one side of the layer is $q(\theta) = \varepsilon_m(\theta)\sigma T_m^4(\theta)$. From the numerical solution the temperature change with time is found throughout the layer and this transient variation is integrated across the layer to obtain the instantaneous heat loss. Then the emittance can be found from the relation

$$\varepsilon_m(\theta) = -\frac{\rho c_p}{2\sigma T_m^4} \int_0^D \frac{\partial T}{\partial \theta}\mathrm{d}x; \quad \varepsilon_m(\tau) = -\frac{2}{t_m^4} \int_0^1 \frac{\partial t}{\partial \tau}\mathrm{d}X \tag{5}$$

To check the numerical accuracy, ε_m was also obtained from the radiative flux, leaving a plane layer as a result of its instantaneous temperature distribution[6]:

$$q_r(\kappa_D, \tau) = 2 \int_0^{\kappa_D} \sigma T^4(\kappa, \tau) E_2(\kappa_D - \kappa)\mathrm{d}\kappa$$

This yields

$$\varepsilon_m(\tau) = \frac{q_r(\kappa_D, \tau)}{\sigma T_m^4} = \frac{2\kappa_D}{t_m^4} \int_0^1 t^4(X, \tau) E_2[\kappa_D(1 - X)]\mathrm{d}X \tag{6}$$

For design purposes, if an approximate expression for $\varepsilon_m(\theta)$ is known by using results such as those given by the present calculations, then the transient mean temperature $t_m(\theta)$ can be calculated by using the heat balance

$$2\varepsilon_m(\theta)\sigma T_m^4(\theta) = -\rho c_p D \frac{\mathrm{d}T_m}{\mathrm{d}\theta} \tag{7}$$

This is integrated with the initial condition $T_m(0) = T_i$, and placed in dimensionless form to yield

$$t_m(\tau) = \left[1 + \frac{3}{2} \int_0^\tau \varepsilon_m(\tau)\mathrm{d}\tau \right]^{-1/3} \tag{8}$$

For comparison with the transient numerical results it is useful to consider results computed by using a transient temperature distribution that remains spatially uniform across the layer. The emittance for this situation is designated as ε_{ut} and is given by[6]

$$\varepsilon_{ut} = 1 - 2E_3(\kappa_D) \tag{9}$$

For $\varepsilon_m(\tau) = \varepsilon_{ut}$, Eq. (8) yields

$$t_{m,ut}(\tau) = \left(1 + \frac{3}{2}\varepsilon_{ut}\tau\right)^{-1/3} \tag{10}$$

Then using $q_{ut}(\tau) = \varepsilon_{ut}\sigma T_i^4 t_{m,ut}^4(\tau)$ the instantaneous energy loss from a layer at uniform temperature is

$$\frac{q_{ut}(\tau)}{\sigma T_i^4} = \varepsilon_{ut}\left(1 + \frac{3}{2}\varepsilon_{ut}\tau\right)^{-4/3} \tag{11}$$

It is evident from Eqs. (9–11) that it is very easy to compute the transient temperature and heat loss for a spatially uniform temperature layer. The numerical results for transient cooling of a layer with spatially nonuniform temperature can be conveniently presented as a ratio relative to the uniform temperature results. The mean temperature $t_m(\tau)$ for the actual transient is then given relative to that for a layer with spatially uniform and time varying temperature by

$$\frac{t_m(\tau)}{t_{m,ut}(\tau, \varepsilon_{ut})} = t_m(\tau)\left(1 + \frac{3}{2}\varepsilon_{ut}\tau\right)^{1/3} \tag{12}$$

Similarly the instantaneous heat loss $q(\tau)$ relative to that for a layer at spatially uniform temperature is

$$\frac{q(\tau)}{q_{ut}(\tau, \varepsilon_{ut})} = \frac{\varepsilon_m(\tau)}{\varepsilon_{ut}} t_m^4(\tau)\left(1 + \frac{3}{2}\varepsilon_{ut}\tau\right)^{4/3} \tag{13}$$

Numerical Solution Procedure

For convenience in developing the solution for Eq. (3) the radiation terms on the right-hand side are called $R(t)$. Then the equation to solve has the form

$$\frac{\partial t}{\partial \tau} = N\frac{\partial^2 t}{\partial X^2} - R(t) \tag{14}$$

Using the trapezoidal rule to integrate over a small time interval, the change in t with τ is expressed in terms of $\partial t/\partial \tau$ by

$$\Delta t = t_{n+1} - t_n = \int_\tau^{\tau+\Delta\tau}\frac{\partial t}{\partial \tau}\,d\tau \approx \frac{\Delta\tau}{2}\left[\left(\frac{\partial t}{\partial \tau}\right)_{n+1} + \left(\frac{\partial t}{\partial \tau}\right)_n\right] \tag{15}$$

The R_{n+1} is expressed in terms of R_n by using the expansion

$$R_{n+1} = R_n + \left(\frac{\partial R}{\partial t}\right)_n (t_{n+1} - t_n) \tag{16}$$

Also the second derivative of t at $\tau + \Delta\tau$ can be written as

$$\left(\frac{\partial^2 t}{\partial X^2}\right)_{n+1} = \frac{\partial^2(t_{n+1} - t_n)}{\partial X^2} + \frac{\partial^2 t_n}{\partial X^2} = \frac{\partial^2 \Delta t_n}{\partial X^2} + \frac{\partial^2 t_n}{\partial X^2} \tag{17}$$

Substituting Eq. (14) into Eq. (15) and then using Eqs. (16) and (17), the equation to be solved for Δt becomes

$$\left[1 + \frac{\Delta\tau}{2}\left(\frac{\partial R}{\partial t}\right)_n - \frac{\Delta\tau}{2}N\frac{\partial^2}{\partial X^2}\right]\Delta t = \Delta\tau\left[N\left(\frac{\partial^2 t}{\partial X^2}\right)_n - R_n\right] \tag{18}$$

To use a variable ΔX increment size across the layer the second derivative is placed in finite difference form with ΔX^- and ΔX^+ extending in the negative and positive directions

$$\frac{\partial^2 t}{\partial X^2} = \frac{2t_{i+1}}{\Delta X^+(\Delta X^+ + \Delta X^-)} - \frac{2t_i}{\Delta X^+\Delta X^-} + \frac{2t_{i-1}}{\Delta X^-(\Delta X^+ + \Delta X^-)} \tag{19}$$

As shown in Fig. 1, there are m locations across the layer so the index $1 \le i \le m$. Since all terms in Eq. (18) are at the time interval corresponding to index n, this subscript will be omitted in what follows and only an i subscript will be used to specify the X location. Thus the relations that follow are for obtaining Δt_i at all grid locations across the layer at time τ_n. This gives the temperatures for all X at the new time by using the relation $t_{n+1} = t_n + \Delta t_n$ at each of the grid locations.

Equation (19) is inserted into Eq. (18) with the result

$$\begin{aligned} &-\frac{\Delta\tau N}{\Delta X^-(\Delta X^+ + \Delta X^-)}\Delta t_{i-1} \\ &+\left(1 + \frac{\Delta\tau}{2}\frac{\partial R_i}{\partial t} + \frac{\Delta\tau N}{\Delta X^+\Delta X^-}\right)\Delta t_i \\ &-\frac{\Delta\tau N}{\Delta X^+(\Delta X^- + \Delta X^+)}\Delta t_{i+1} \\ &= \Delta\tau\left(\frac{2N}{\Delta X^- + \Delta X^+}\left[\frac{t_{i+1}}{\Delta X^+} - \left(\frac{\Delta X^+ + \Delta X^-}{\Delta X^+\Delta X^-}\right)t_i + \frac{t_{i-1}}{\Delta X^-}\right] - R_i\right) \end{aligned} \tag{20a}$$

where $\Delta X^+ = X_{i+1} - X_i$ and $\Delta X^- = X_i - X_{i-1}$. Equation (20a) is valid for all interior points $2 \le i \le m - 1$. At each boundary there is a zero normal temperature derivative as explained earlier. Hence Eq. (20) has a special form at the boundaries obtained by letting the temperature at a mirror image grid point be the same as the value at the first grid point away from the boundary. Thus at $i = 1$, Eq. (20) is modified by letting the value at a fictitious point $i = 0$ be $t_0 = t_2$, and by letting $\Delta X^- = \Delta X^+$. This yields at $i = 1$

$$\begin{aligned} &\left[1 + \frac{\Delta\tau}{2}\left(\frac{\partial R_1}{\partial t}\right) + \frac{\Delta\tau N}{(\Delta X^+)^2}\right]\Delta t_1 - \frac{\Delta\tau N}{(\Delta X^+)^2}\Delta t_2 \\ &= \Delta\tau\left[\frac{2N}{(\Delta X^+)^2}(t_2 - t_1) - R_1\right] \end{aligned} \tag{20b}$$

Similarly at $i = m$

$$\begin{aligned} &-\frac{\Delta\tau N}{(\Delta X^-)^2}\Delta t_{m-1} + \left[1 + \frac{\Delta\tau}{2}\frac{\partial R_m}{\partial t} + \frac{\Delta\tau N}{(\Delta X^-)^2}\right]\Delta t_m \\ &= \Delta t\left[\frac{2N}{(\Delta X^-)^2}(t_{m-1} - t_m) - R_m\right] \end{aligned} \tag{20c}$$

Equations (20a)–(20c) are assembled into a tridiagonal matrix to yield

$$\begin{bmatrix} b_1 & c_1 & & \\ a_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{m-1} \\ & & a_m & b_m \end{bmatrix}\begin{bmatrix} \Delta t_1 \\ \Delta t_2 \\ \vdots \\ \Delta t_m \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} \tag{21}$$

where

$$a_j = -\frac{\Delta\tau N}{\Delta X^-(\Delta X^+ + \Delta X^-)}, \quad 2 \le j \le m-1;$$

$$a_m = -\frac{\Delta\tau N}{(\Delta X^-)^2}$$

$$b_1 = 1 + \frac{\Delta\tau}{2}\frac{\partial R_1}{\partial t} + \frac{\Delta\tau N}{(\Delta X^+)^2};$$

$$b_j = 1 + \frac{\Delta\tau}{2}\frac{\partial R_j}{\partial t} + \frac{\Delta\tau N}{\Delta X^+ \Delta X^-}, \qquad 2 \le j \le m-1;$$

$$b_m = 1 + \frac{\Delta\tau}{2}\frac{\partial R_m}{\partial t} + \frac{\Delta\tau N}{\Delta X^-)^2}$$

$$c_1 = -\frac{\Delta\tau N}{(\Delta X^+)^2};$$

$$c_j = -\frac{\Delta\tau N}{\Delta X^+(\Delta X^+ + \Delta X^-)}, \qquad 2 \le j \le m-1$$

$$s_1 = \Delta\tau\left[\frac{2N}{(\Delta X^+)^2}(t_2 - t_1) - R_1\right];$$

$$s_j = \Delta\tau\left\{\frac{2N}{\Delta X^+ + \Delta X^-}\left[\frac{t_{j+1}}{\Delta X^+} - \left(\frac{\Delta X^+ + \Delta X^-}{\Delta X^+ \Delta X^-}\right)t_j + \frac{t_{j-1}}{\Delta X^-}\right] - R_j\right\}, 2 \le j \le m-1;$$

$$s_m = \Delta\tau\left[\frac{2N}{(\Delta X^-)^2}(t_{m-1} - t_m) - R_m\right]$$

From $R(t)$ being defined as the radiation terms in Eq. (3), the $\partial R/\partial t$ needed for the b coefficients is given by

$$\frac{\partial R}{\partial t} = 4\kappa_D\left[t^3 - \frac{\kappa_D}{2}\int_0^1 t^3 E_1(|X - X^*|)\mathrm{d}X^*\right] \tag{22}$$

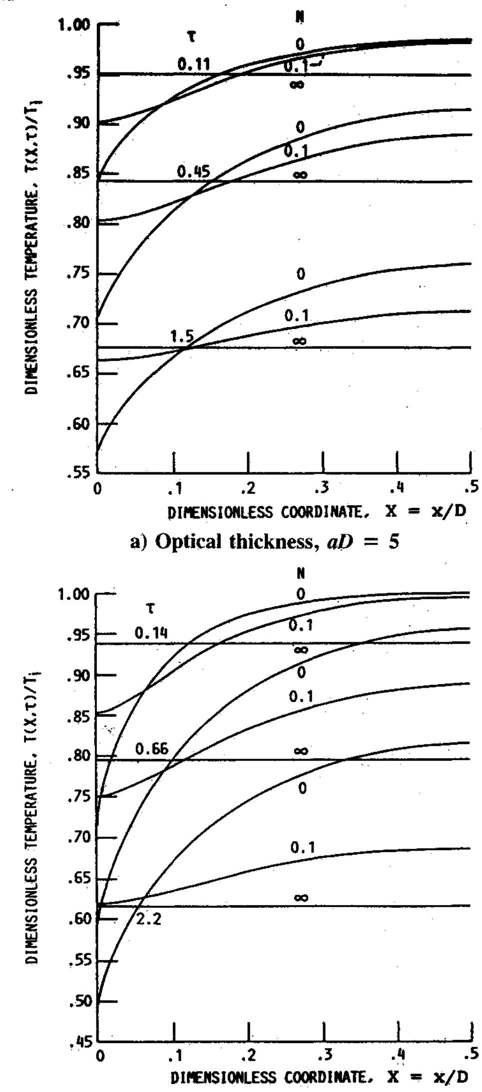
The tridiagonal matrix, Eq. (21), is solved using the well-known algorithm as in Refs. 7 and 8. The Δt at each X is then added to the t values to advance to the next time increment.

To evaluate the radiative source term $R(X)$ and its derivative that are in the matrix coefficients, an accurate integration method is required. Since $E_1(0) = \infty$ care is necessary as X^* approaches X. The integral of E_1 is $-E_2$, and $E_2(0) = 1$, so the integration is carried out analytically for a very small region near X with t^3 or t^4 kept constant at their values at X. This was shown to provide accurate results in a previous analysis.[4] The integrations in the regions away from the singularity were performed with an available Gaussian integration subroutine. Values of the functions at the unevenly spaced Gaussian points needed for the subroutine were interpolated from the values at the specified grid points by using cubic spline curve fitting and interpolation. By using various numbers of grid locations, it was found that 50 ΔX increments across the layer gave accurate results. The increment size was small near the boundaries; 10 increments with $\Delta X = 0.01$ were used adjacent to each boundary. A variable time increment was used with $\Delta\tau = 0.01$ initially, and then gradually increasing throughout the calculation as the rate of temperature change decreased. The time increments were such that $t_m(\tau)$ changed about 0.01 for each time increment.

Results and Discussion

Transient Temperature Distributions

The numerical solution yields transient temperature distributions. Typical results are in Figs. 2a and 2b for optical thicknesses, $\kappa_D = 5$ and 15. From symmetry, only half the layer is shown. When the transient begins, the layer is suddenly subjected to cold surroundings, and the outer portions cool rapidly. If the optical thickness is large such as in Fig. 2b, the temperature profile can vary rapidly with X near each boundary if $N = 0$ or is small ($N < 0.01$). When there is finite heat conduction, $N > 0$, the conditions of the present problem require a zero normal temperature gradient at the boundaries (the gradient is finite when $N = 0$). For $N = 0.1$, heat conduction provides a significant equalization of the temperature distribution across the layer.



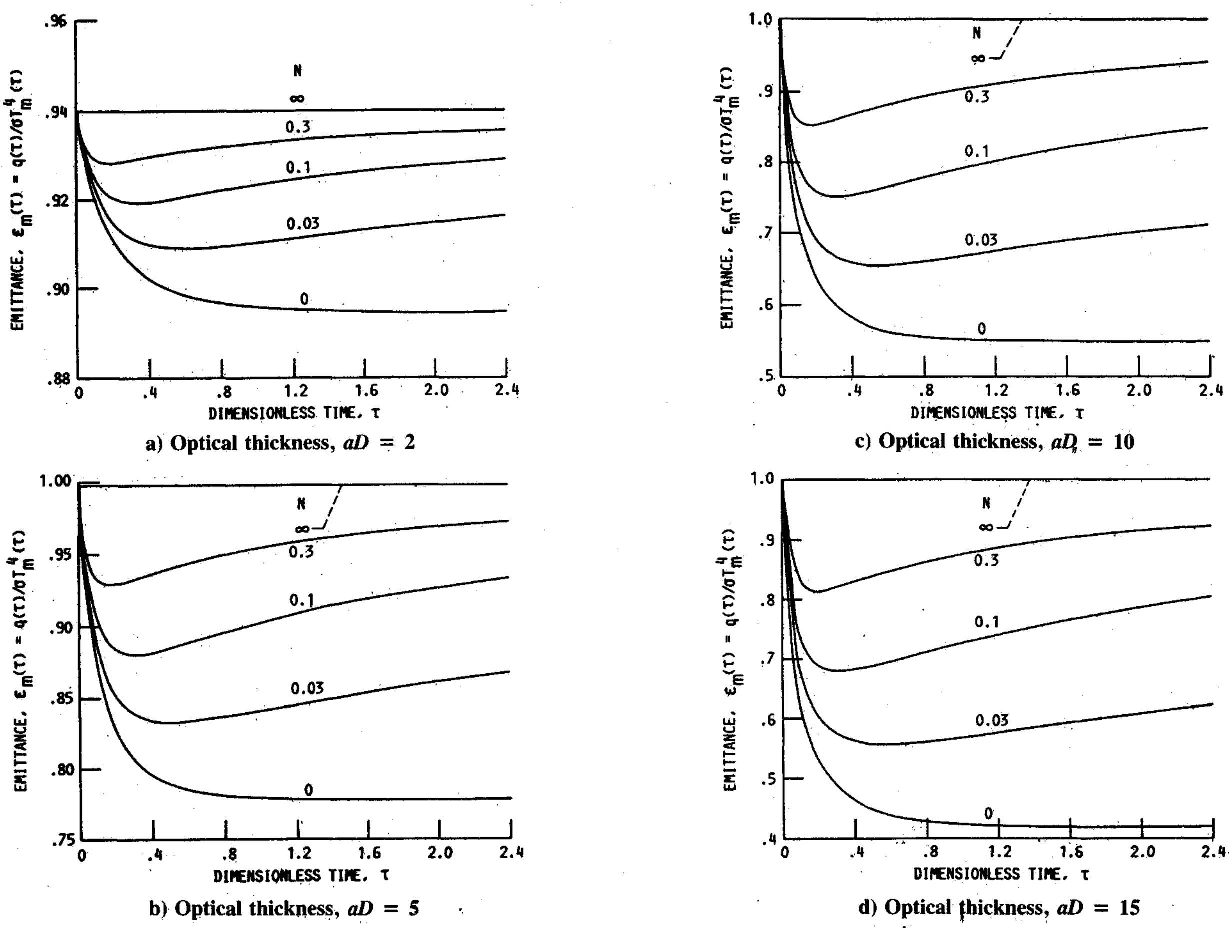
a) Optical thickness, $aD = 5$

b) Optical thickness, $aD = 15$

Fig. 2 Effect of heat conduction on transient temperature distributions for two optical thicknesses.

Transient Emittance

Using the temperature distributions, the transient emittance is calculated from Eqs. (5) and (6). Results for $\kappa_D = 2$, 5, 10, and 15 and various N values are in Figs. 3a–3d. For $N = 0$ there is radiation only. When conduction is very large compared with radiation, $N \to \infty$, conduction provides a uniform temperature distribution as the energy is being radiated away. Then the emittance remains at the ε_{ut} value throughout the transient. The transient emittance is the instantaneous heat flux radiated from one side of the layer divided by $\sigma T_m^4(\tau)$. Since the layer is initially at uniform temperature, the emittance is initially $\varepsilon_m(0) = \varepsilon_{ut} = 1 - 2E_3(\kappa_D)$. The ε_{ut} values corresponding to $\kappa_D = 2$, 5, 10, and 15 are 0.940, 0.998, 1.000, and 1.000. The curves in Figs. 3a–3d start at these values. As the transient begins, the outer regions of the



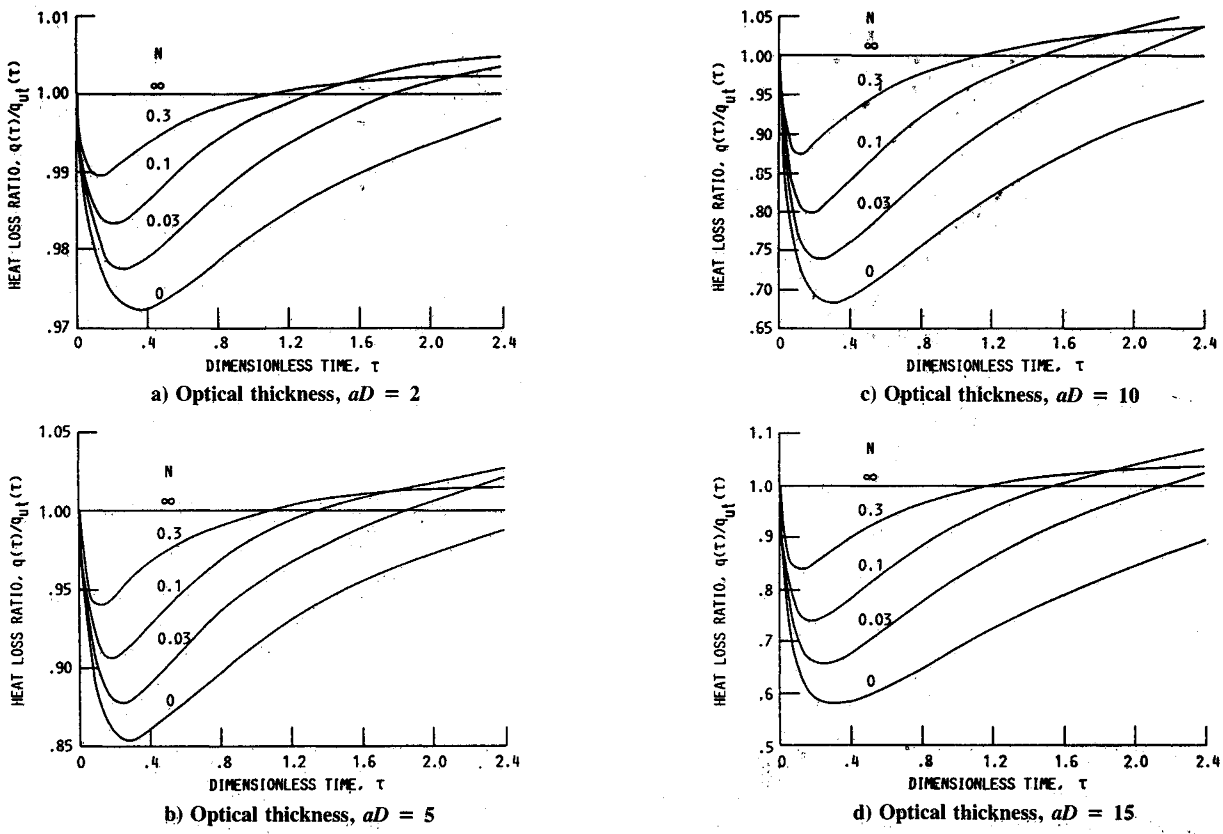
a) Optical thickness, $aD = 2$

b) Optical thickness, $aD = 5$

c) Optical thickness, $aD = 10$

d) Optical thickness, $aD = 15$

Fig. 3 Effect of radiation-conduction parameter on transient emittance of layer as a function of optical thickness.



a) Optical thickness, $aD = 2$

b) Optical thickness, $aD = 5$

c) Optical thickness, $aD = 10$

d) Optical thickness, $aD = 15$

Fig. 4 Transient heat loss as compared with that for a layer cooling with a uniform instantaneous temperature distribution.

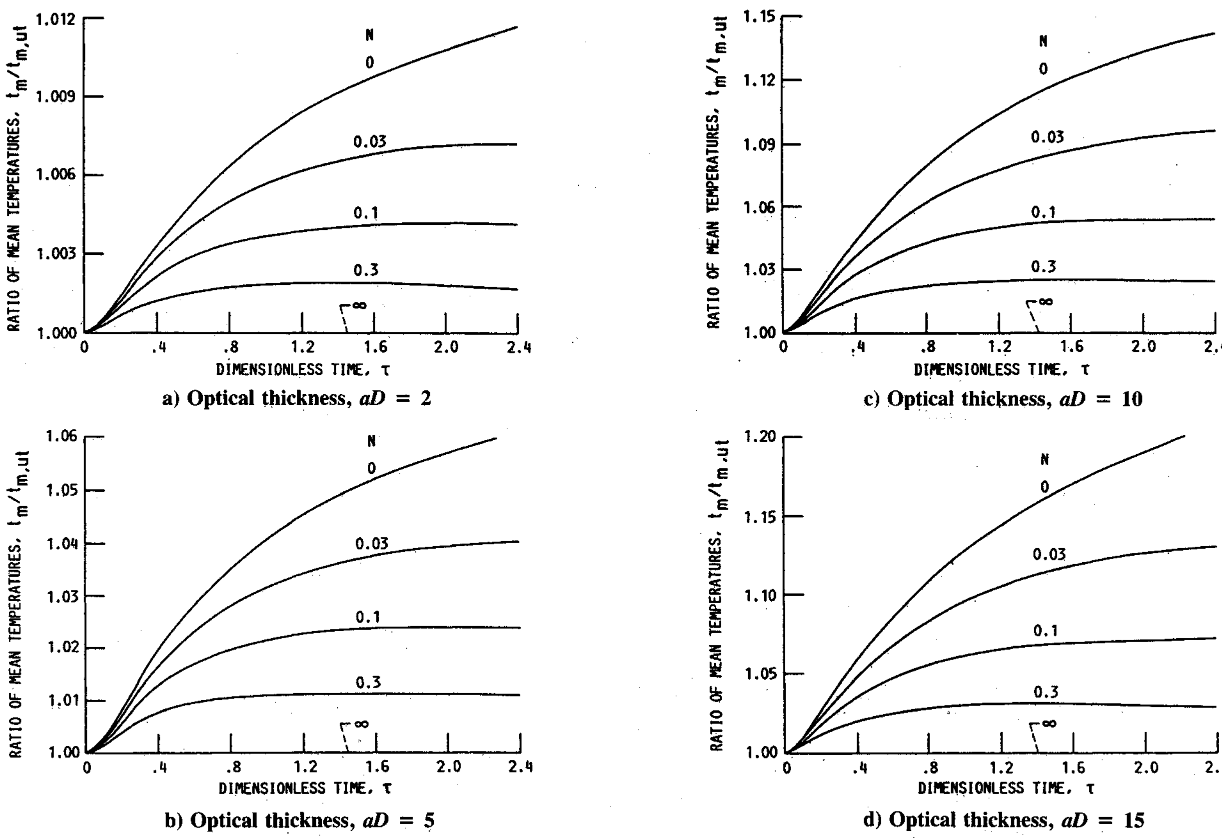


Fig. 5 Mean temperature of layer during transient cooling as compared with that for a layer cooling with a uniform instantaneous temperature distribution.

layer cool rapidly and emission decreases rapidly so that the radiant energy being lost is smaller than that characteristic of the instantaneous mean temperature. As shown by the similarity solution in Ref. 5, for the case with no heat conduction, the transient emittance decreases to a steady value that depends on the optical thickness of the layer. With conduction present, the temperature distribution gradually becomes more uniform as the transient proceeds because the decreasing temperature level causes radiation to become less important relative to conduction. As the temperature distribution becomes more uniform, the transient emittance rises toward its initial value corresponding to a uniform temperature layer. This is an asymptotic approach, but the behavior is evident from the results shown.

Transient Heat Loss

The quantity in Figs. 4a–4d is the instantaneous heat loss from the layer. This is given as a ratio to the values for a spatially uniform, but time varying, temperature as obtained from Eq. (11). Since values from Eq. (11) are easily calculated, the $q(\tau)$ can be readily obtained for design purposes from the ratios given. During the early portion of the transient the rate of energy loss is lower than that for a layer with $t(\tau)$ independent of X. As time proceeds, the ratios eventually become a little larger than unity. This is because the mean temperature has decreased more slowly than for the uniform temperature case (shown quantitatively in Fig. 5). As a result, late in the transient the mean temperature is large enough that the heat loss is above that reached for the spatially uniform temperature case at that time. This effect is not of much practical significance because late in the transient the temperatures are small and the radiative losses have become very small.

Transient Mean Temperature

With regard to the total amount of energy that has been radiated away, the quantity of interest is the mean temperature as a function of time. This is in Fig. 5 for various κ_D. The ordinate is the ratio of the instantaneous mean temperature to the value that would be reached at that time if the layer always had $t(\tau)$ independent of X and hence had the emittance given by Eq. (9). The ratio is unity for $N \to \infty$ since the temperature distribution becomes uniform when conduction is very large relative to radiation. For any finite N the $t_m(\tau)$ at any time is larger than the value computed from the uniform temperature relations, and the size of the ratio increases with optical thickness. For design use, the mean temperature $t_m(\tau)$ for various aD can be obtained by using Eq. (10) in conjunction with the ratios given on the curves. For a small optical thickness the layer always remains at a fairly uniform temperature. For an optical thickness of $aD = 2$, Fig. 5a, the instantaneous mean temperature is at most only about 1% above the spatially uniform temperature value, so for any optical thickness less than 2 the simple Eq. (10) can be used to obtain a very good approximation to the instantaneous mean temperature for any N. When $N > 0.1$, Fig. 5b shows that agreement with Eq. (10) within 2% is obtained when κ_D is 5 or less. Similarly for $N > 0.3$ agreement within a few percent is obtained for aD as large as 15.

Concluding Remarks

Transient solutions were obtained for a radiating and conducting layer that is being cooled by exposure to a cold vacuum environment. Because of the boundary conditions, and the large temperature gradients near the boundaries for some conditions, approximate methods based on diffusion or moment expansion methods are often not sufficiently accurate. It is demonstrated that direct numerical techniques can be used without difficulty to obtain accurate transient temperature distributions. Computer times on a CRAY-XMP are about 1 min for a complete transient solution. A combination was used of Gaussian integration and an implicit finite difference method with variable spatial and time increments.

Transient results for the layer mean temperature were compared with a simple analytical expression for a layer with a spatially uniform temperature at any instant during the transient. The ranges are provided, of optical thickness and conduction parameter, within which the simplified theory will provide transient mean temperature results that are accurate within a few percent.

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